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On a Class of Solutions of Nonlinear Boltzmann Equations

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In a recent paper by Krook and Wu, the nonlinear Boltzmann equation for an infinite, spatially homogeneous, isotropic monoatomic gas of constant density and kinetic energy and with an elastic differential cross section that varies inversely as relative speed has been reduced to an infinite sequence of moment equations. The present note observes that the moment equations are successively integrable and shows that as time goes to infinity, the distribution tends to be Maxwellian.

KEY WORDS: Boltzmann equation; special cross section; exact solution; moment equations.

1. INTRODUCTION

The state of the gas at time t is described by a distribution function $nf(v, t)$, where n is the constant number density, $\mathbf{v}$ is a velocity variable, and $v = |\mathbf{v}|$. Conservation of mass and energy imply that

$$\int f(v, t)\, d^3v = 1, \qquad \int v^2 f(v, t)\, d^3v = 3\kappa T/m = 3\beta^2 \tag{1}$$

where T is the constant kinetic temperature, m is the molecular mass, and κ is Boltzmann's constant.

The differential cross section for elastic scattering is taken as

$$\sigma(g, \chi) = K/g \tag{2}$$

where K is a constant, g is the relative speed, and χ is the scattering angle in the center-of-mass system.

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Then the nonlinear Boltzmann equation is equivalent to the following sequence of moment equations[1]:

$$\frac{dM_k}{d\tau} + M_k = \frac{1}{k+1} \sum_{m=0}^{k} M_m M_{k-m}, \qquad k = 0, 1, 2, \ldots \tag{3}$$

where

$$M_k = \frac{\sqrt{\pi}}{2(2\beta^2)^k \Gamma(k + \frac{3}{2})} \int v^{2k} f(v, \tau)\, d^3v, \qquad k = 0, 1, 2, \ldots \tag{4}$$

M_k is called the kth moment and

$$\tau = 4\pi nkt \tag{5}$$

Using (4), one can replace (1) by

$$M_0(\tau) = 1, \qquad M_1(\tau) = 1 \tag{6}$$

If, as $\tau \to \infty$, the distribution tends to be Maxwellian, then we must have

$$M_k(\infty) = 1 \qquad \text{for} \quad k = 0, 1, 2, \ldots \tag{7}$$

For given (6), Krook and Wu[1] have obtained some particular solutions of (3), satisfying the "boundary conditions" (7). From these solutions of (3), the distribution function $f(v, \tau)$ has been obtained by inverting (4). However, in the present note we shall see that the complete set of solutions of (3) that satisfy (6) can be obtained by successive integration and that Eqs. (7) follow from (3) and (6) and need not be postulated as "boundary conditions."

2. SOLUTIONS OF THE MOMENT EQUATIONS

Equations (3) can be explicitly written as

$$\begin{aligned} dM_0/d\tau + M_0 &= M_0^2 \\ dM_1/d\tau + M_1 &= M_0 M_1 \cdot \\ dM_2/d\tau + M_2 &= \tfrac{1}{3}(2M_0M_2 + M_1^2) \\ dM_3/d\tau + M_3 &= \tfrac{1}{2}(M_0M_3 + M_1M_2) \\ dM_4/d\tau + M_4 &= \tfrac{1}{5}(2M_0M_4 + 2M_1M_3 + M_2^2) \end{aligned} \tag{8}$$

and so on.

We note that the first two equations of (8) are automatically satisfied by (6). All other moment equations can be written as

$$\begin{aligned} \frac{dM_k}{d\tau} + M_k &= \frac{1}{k+1}(2M_0M_k + 2M_1M_{k-1} + 2M_2M_{k-2} \\ &\quad + \cdots + 2M_{(k-1)/2}M_{(k+1)/2} \qquad \text{when } k \text{ is odd and } \geqslant 2 \\ \frac{dM_k}{d\tau} + M_k &= \frac{1}{k+1}(2M_0M_k + 2M_1M_{k-1} + 2M_2M_{k-2} \\ &\quad + \cdots + 2M_{(k/2)-1}M_{(k/2)+1} + M_{k/2}^2) \qquad \text{when } k \text{ is even and } \geqslant 2 \end{aligned} \tag{9}$$

Using (6), one can rewrite (9) as

$$\frac{dM_k}{d\tau} + \frac{k-1}{k+1} M_k = \frac{1}{k+1}(2M_{k-1} + 2M_2M_{k-2} + \cdots + 2M_{(k-1)/2}M_{(k+1)/2})$$

when k is odd and $\geqslant 2$

$$\frac{dM_k}{d\tau} + \frac{k-1}{k+1} M_k = \frac{1}{k+1}(2M_{k-1} + 2M_2M_{k-2} + \cdots + 2M_{(k/2)-1}M_{(k/2)+1} + M_{k/2}^2) \tag{10}$$

when k is even and $\geqslant 2$

Equations (10) have the following solution:

$$M_k = A_k e^{-[(k-1)/(k+1)]\tau} + \frac{e^{-[(k-1)/(k+1)]\tau}}{k+1} \int e^{[(k-1)/(k+1)]\tau}(2M_{k-1} + 2M_2M_{k-2} + \cdots + 2M_{(k-1)/2}M_{(k+1)/2})\, d\tau \quad \text{when } k \text{ is odd and } \geqslant 2 \tag{11}$$

$$M_k = A_k e^{-[(k-1)/(k+1)]\tau} + \frac{e^{-[(k-1)/(k+1)]\tau}}{k+1} \int e^{[(k-1)/(k+1)]\tau}(2M_{k-1} + 2M_2M_{k-2} + \cdots + 2M_{(k/2)-1}M_{(k)/2+1} + M_{k/2}^2)\, d\tau \quad \text{when } k \text{ is even and } \geqslant 2$$

where A_k ($k = 2, 3, 4, 5,\ldots$) are constants.

We note that for any $k \geqslant 2$, M_k can be determined from (11) if all of $M_{k-1}, M_{k-2},\ldots$ are known. Thus M_2 can be determined since M_0 and M_1 are given by (6). Similarly, M_3 can be determined from M_0, M_1, and M_2, and so on. The constants of integration A_k can be determined if $M_k(0)$, $k = 0, 1, 2, 3,$ are known.

The first few moments can be written as

$$M_0 = 1, \quad M_1 = 1, \quad M_2 = 1 + A_2e^{-\tau/3}, \quad M_3 = 1 + 3A_2e^{-\tau/3} + A_3e^{-\tau/2}$$
$$M_4 = 1 + 6A_2e^{-\tau/3} + 4A_3e^{-\tau/2} - 2A_2{}^2e^{-2\tau/3} + A_4e^{-3\tau/5} \tag{12}$$

From Eqs. (12), we note that all of the first five moments are of the form

$$M_k = 1 + \tfrac{1}{2}k(k-1)A_2e^{-\tau/3} + \tfrac{1}{6}k(k-1)(k-2)A_3e^{-\tau/2} + \sum_r B_{kr}e^{-a_{kr}\tau} \tag{13}$$

where A_2, A_3, B_{kr}, a_{kr} are constants and $a_{kr} > \frac{1}{2}$. We shall show that in fact all the moments are of the form (13). The proof is by mathematical induction.

Let the M_k for $k = l, l-1, l-2,\ldots$ be of the form (13). We shall show that M_{l+1} is also of the form (13). From (11) we get

$$M_{l+1} = A_{l+1}e^{-l\tau/(l+2)} + \frac{e^{-l\tau/(l+2)}}{l+2} \int e^{l\tau(l+2)} I\, d\tau \tag{14}$$

where

$$I = 2M_l + 2M_2M_{l-1} + 2M_3M_{l-2} + \cdots + 2M_{l/2}M_{(l/2)+1}, \qquad l \text{ even}$$
$$= 2M_l + 2M_2M_{l-1} + 2M_3M_{l-2} + \cdots + 2M_{(l-1)/2}M_{(l+3)/2} + M^2_{(l+1)/2}, \qquad l \text{ odd} \tag{15}$$

If $M_l, M_{l-1}, M_{l-2}, \ldots$ are of the form (13), then it is easy to see from (15) that I is of the form

$$I = l + 2A_2e^{-\tau/3}\left(\sum_{r=2}^{l} \frac{r(r-1)}{2}\right) + 2A_3e^{-\tau/2}\left(\sum_{r=3}^{l} \frac{r(r-1)(r-2)}{6}\right) + \sum_{r} D_{l+1,r}e^{-a_{l+1,r}\tau} \tag{16}$$

where A_2, A_3, $D_{l+1,r}$, $a_{l+1,r}$ are constants and $a_{l+1,r} > \frac{1}{2}$. From (14) and (16) one can easily see that M_{l+1} is of the form (13).

Thus M_{l+1} is of the form (13) if $M_l, M_{l-1}, M_{l-2}, \ldots$ are of the form (13) and we have already seen that M_0, M_1, M_2, M_3, and M_4 are of the form (13). Thus the M_k for $k = 0, 1, 2, 3, \ldots$ are all of the form (13).

3. CONCLUSION

Thus, although we have not been able to give all the moments explicitly, we have a set of equations [Eqs. (6) and (11)] from which all the moments can be successively determined. We also know that all the moments are of the form (13), from which we note that as $\tau \to \infty$, $M_k \to 1$ for $k = 0, 1, 2, \ldots$; which means that as time goes to infinity the distribution tends to be Maxwellian.

As pointed out by Krook and Wu,[(1)] knowledge of the nature of $f(\mathbf{v}, t)$ for large values of v is of utmost importance, because large deviation of $f(\mathbf{v}, t)$ from the Maxwellian distribution for large values of v can significantly alter the calculated values of certain gas reaction rates. The moments calculated in the present work can be used to put an upper limit on the number of particles that exceed any given velocity. To this end, from Tchebycheff's theorem,[(2)] we get for any $v_0 > 0$,[3]

$$P(v^{2r} > v_0^{2r}) \leqslant E(v^{2r})/v_0^{2r}), \qquad r = 1, 2, 3, \ldots \tag{I}$$

where $P(x > \alpha) \equiv$ Probability of $x > \alpha$,

$$E(g(\mathbf{v}, t)) \equiv \int g(\mathbf{v}, t) f(\mathbf{v}, t)\, d^3\mathbf{v}$$

From (4) and (I) and noting that $P(v > v_0) = P(v^{2r} > v_0^{2r})$, we get

$$P(v > v_0) \leqslant \frac{2(2\beta^2)^r\Gamma(r + \frac{3}{2})M_r}{\sqrt{z}\, v_0^{2r}}, \qquad r = 1, 2, 3, \ldots \tag{II}$$

[3] Roman numerals are used to denote inequalities.

The inequality (II) for each value of r sets an upper limit on $P(v > v_0)$. Obviously the value of r that gives the least value for the right-hand side of (II) also gives the most accurate upper limit for $P(v > v_0)$. But since we do not have explicit expressions for all the moments, we are not in a position to know which value of r will give the most accurate upper limit. However, we can take a few values of r to see which one serves our purposes better; e.g., $r = 1$ gives

$$P(v > v_0) \leqslant 3\beta^2/v_0^2 \tag{III}$$

$r = 2$ gives

$$P(v > v_0) \leqslant (15\beta^4/v_0^4)(1 + A_2 e^{-\tau/3}) \tag{IV}$$

For $A_2 < 0$ (which is equivalent to saying that $M_2 < 1$ at $\tau = 0$) and $v_0^2 > 5\beta^2$ then the inequality (IV) gives the more accurate upper limit. On the other hand, if $A_2 > 0$ and $v_0^2 < 5\beta^2$, then it would be better to use (III). Similar inferences can be drawn for other values of r as well, but things will be increasingly more complicated.

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